# CSCE 2110
# Foundations of Data Structures

Sorting

# Content

- Comparison-based sorting algorithms:
  - Insertion sort
  - Selection sort
  - Heapsort
  - Merge sort
  - Quick sort

- Integer sorting (optional):
  - Bucket sort
  - Radix sort

# Sorting

- Given a set (container) of $n$ elements:
  - e.g., array, set of words, etc.
- Suppose there is an order relation that can be set across the elements
- Goal: Arrange the elements in a certain order
  - e.g., ascending/descending orders

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- Given a set (container) of $n$ elements:
  - e.g., array, set of words, etc.
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Before sorting: 1 23 2 56 9 8 10 100

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- Goal: Arrange the elements in a certain order
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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Before sorting: | 1 | 23 | 2 | 56 | 9 | 8 | 10 | 100 |
| After sorting: | 1 | 2 | 8 | 9 | 10 | 23 | 56 | 100 |

# Importance of Sorting

- Why don't CS profs ever stop talking about sorting?
  - Computers spend a lot of time sorting, historically 25% on mainframes
  - Sorting is the best studied problem in computer science, with many different algorithms known
  - Most of the interesting ideas we will encounter in the course can be taught in the context of sorting, such as divide-and-conquer, randomized algorithms, and lower bounds

# Stable Sorting

- A property of sorting
- If a sort guarantees the relative order of equal items stays the same, then it is a stable sort

Before sorting: $7_1, 6, 7_2, 5, 1, 2, 7_3, -5$ (subscripts added for clarity)

After sorting: $-5, 1, 2, 5, 6, 7_1, 7_2, 7_3$ (result of stable sort)

# In Place Sorting

- Sorting of a data structure does not require any external data  structure for storing the intermediate steps
- The amount of extra space required to sort the data is constant with the input size

# Insertion Sort

- Insertion sort: orders a list of values by repetitively inserting a particular value into a sorted subset of the list
- More specifically:

1) consider the first item to be a sorted sub list of length 1
2) insert the second item into the sorted sub list, shifting the first item if needed
3) insert the third item into the sorted sub list, shifting the other items as needed
4) repeat until all values have been inserted into their proper positions

# Insertion Sort

```
template <class Item>
void insertion_sort(Item data[ ], size_t n) {
     size_t i, j;
     Item temp;

     if(n < 2) return;  // nothing to sort!!

     for(i = 1; i < n; ++i)
     {
       // take next item at front of unsorted part of array
       // and insert it in appropriate location in sorted part of array
       temp = data[i];
       for(j = i; data[j-1] > temp and j > 0; --j)
          data[j] = data[j-1];  // shift element forward

       data[j] = temp;
      }
}
```

# Insertion Sort: Example

- Sorting: 3, 9, 6, 1, 2 using insertion sort

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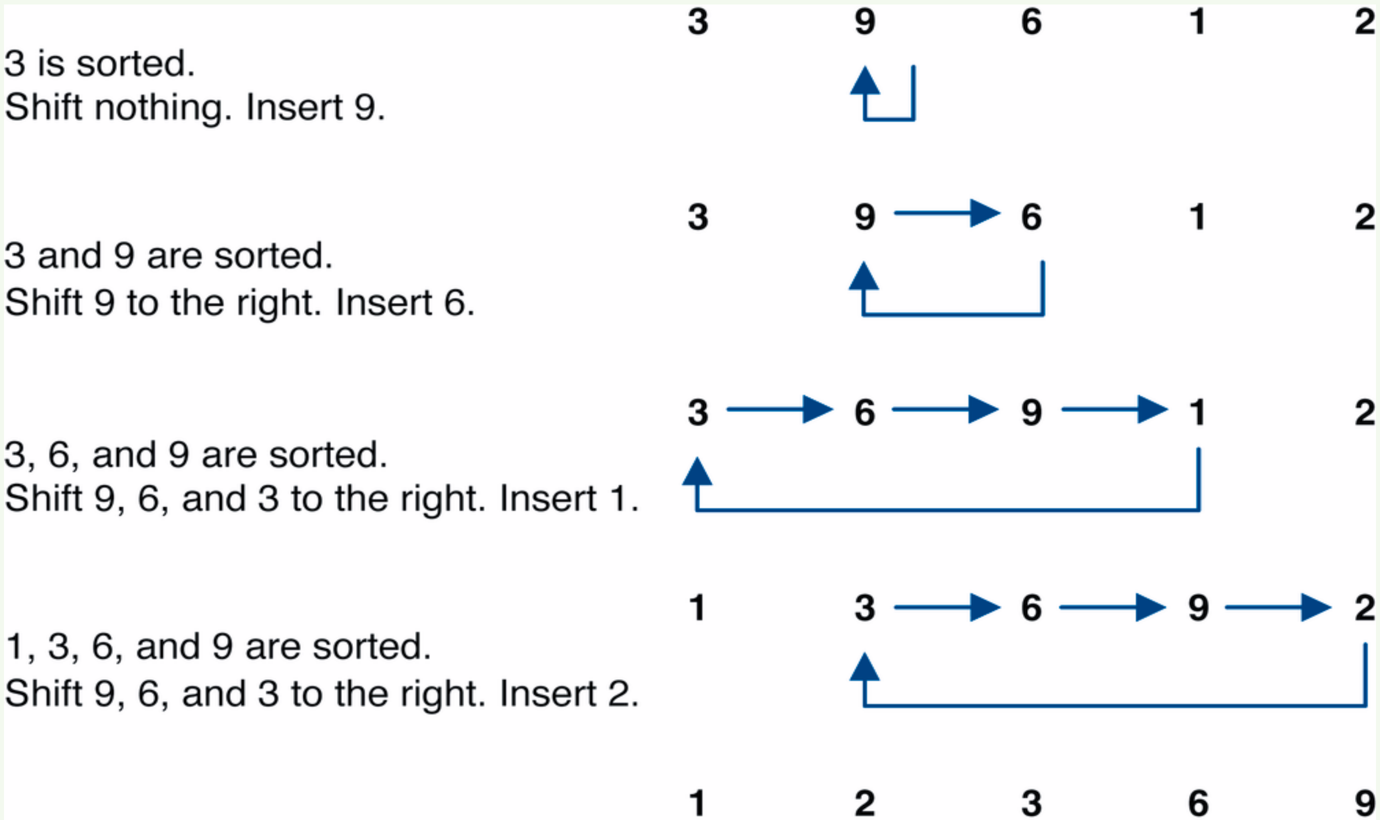

# Insertion Sort Time Analysis

- In $O$-notation, what is:
  - Worst case running time for $n$ items?
  - Best case running time for $n$ items?

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  - Worst case running time for $n$ items?
  - Best case running time for $n$ items?

- Steps of algorithm:

```
for i = 1 to n-1
   take next key from unsorted part of array
   insert in appropriate location in sorted part of array:
      for j = i down to 0,
            shift sorted elements to the right if key > key[i]
   increase size of sorted array by 1
```

Outer loop: O(n)

Inner loop: O(n)

# Selection Sort

- Basic idea:
  1) Find the smallest element in the array
  2) Exchange it with the element in the first position
  3) Find the second smallest element and exchange it with the element in the second position
  4) Continue until the array is sorted

# Selection Sort

```
SELECTION-SORT(A):
    n ← length[A]
    for j ← 1 to n - 1
        do smallest ← j
           for i ← j + 1 to n
               do if A[i] < A[smallest]
                   then smallest ← i
           exchange A[j] ↔ A[smallest]
```

# Selection Sort: Example

- Sorting: 8, 4, 6, 9, 2, 3, 1 using selection sort

# Selection Sort: Example

- Sorting: 8, 4, 6, 9, 2, 3, 1 using insertion sort

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 4 | 6 | 9 | 2 | 3 | 1 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 9 | 2 | 3 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 6 | 9 | 4 | 3 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 9 | 4 | 6 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 9 | 6 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 9 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 9 |

# Selection Sort Time Analysis

- It's clearly quadratic:
  - The first pass, we search through exactly $n - 1$ elements (no difference between average-case and worst-case), then swap (constant time)
  - Second time, $n - 2$ elements, then $n - 3$, etc.

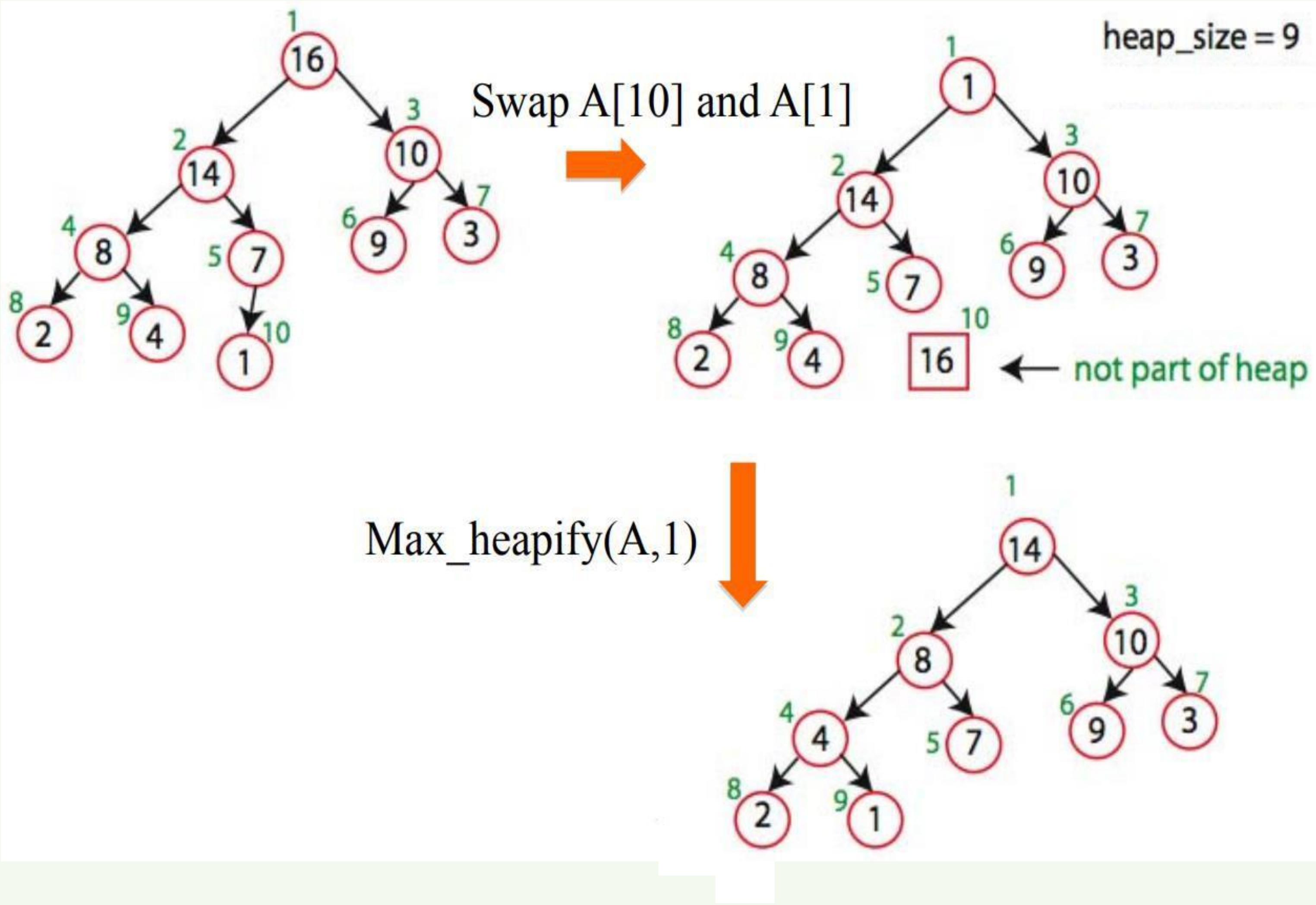
We get the arithmetic sum $(n-1)+(n-2)+(n-3)+\ldots+1= O(n^2)$

# Heapsort

## Sorting Strategy:

- Build Max Heap from unordered array;
- Find maximum element $A[1]$;
- Swap elements $A[n]$ and $A[1]$: now max element is at the end of the array!
- Discard node $n$ from heap (by decrementing heap-size variable)
- New root may violate max heap property, but its children are max heaps. Run `max_heapify` to fix this.
- Go to Step 2 unless heap is empty.

# Heapsort Demo



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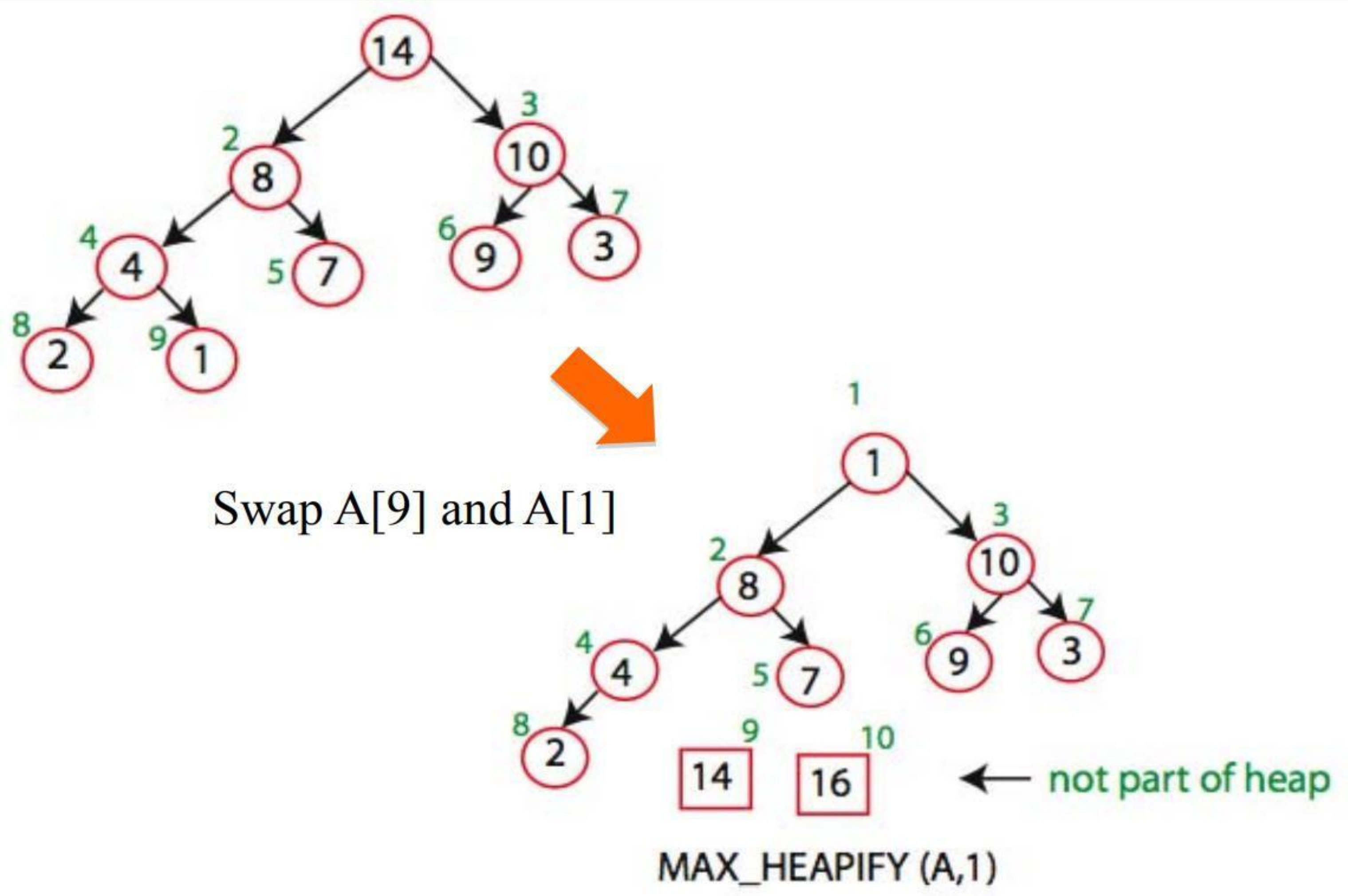

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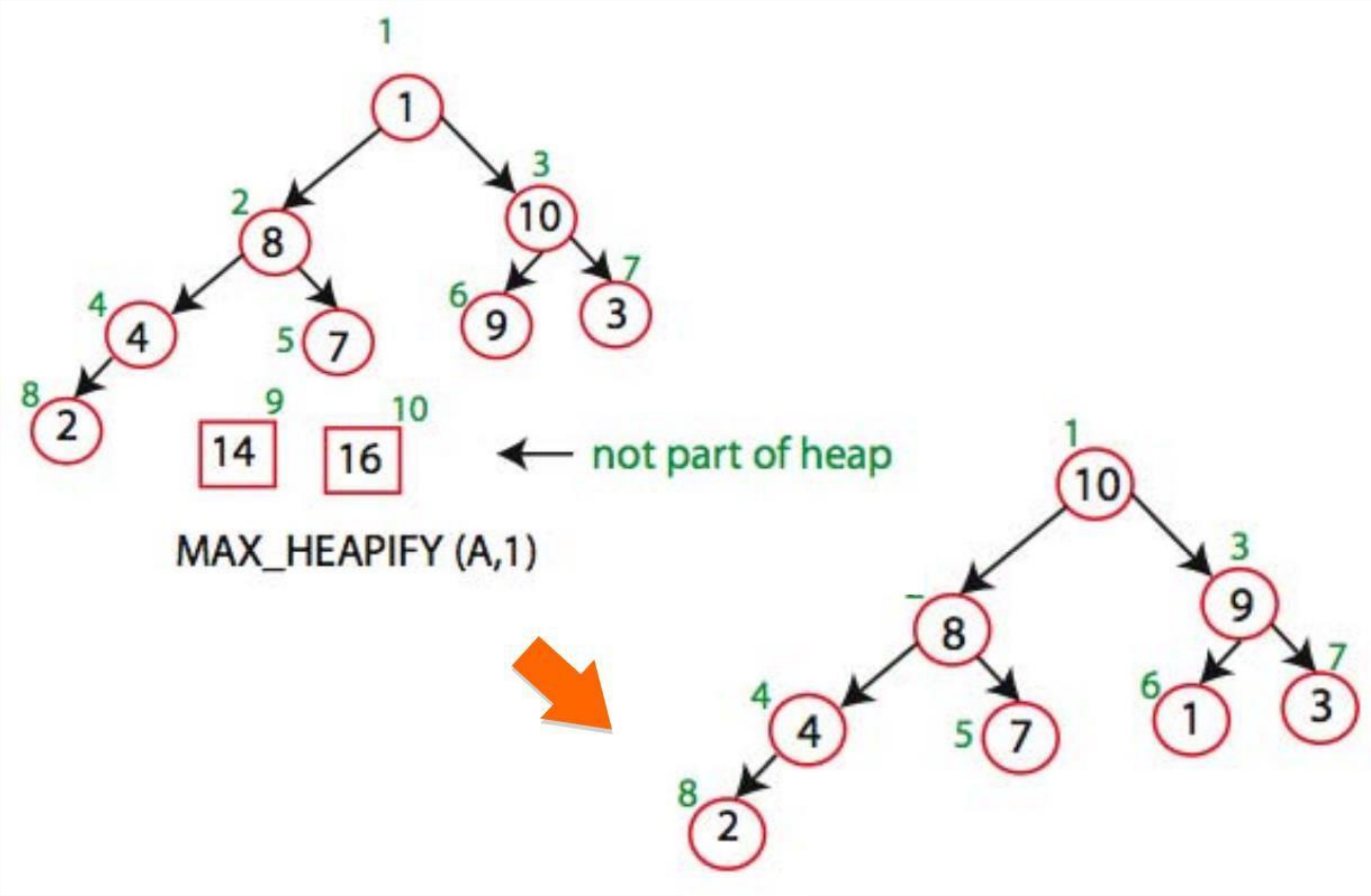

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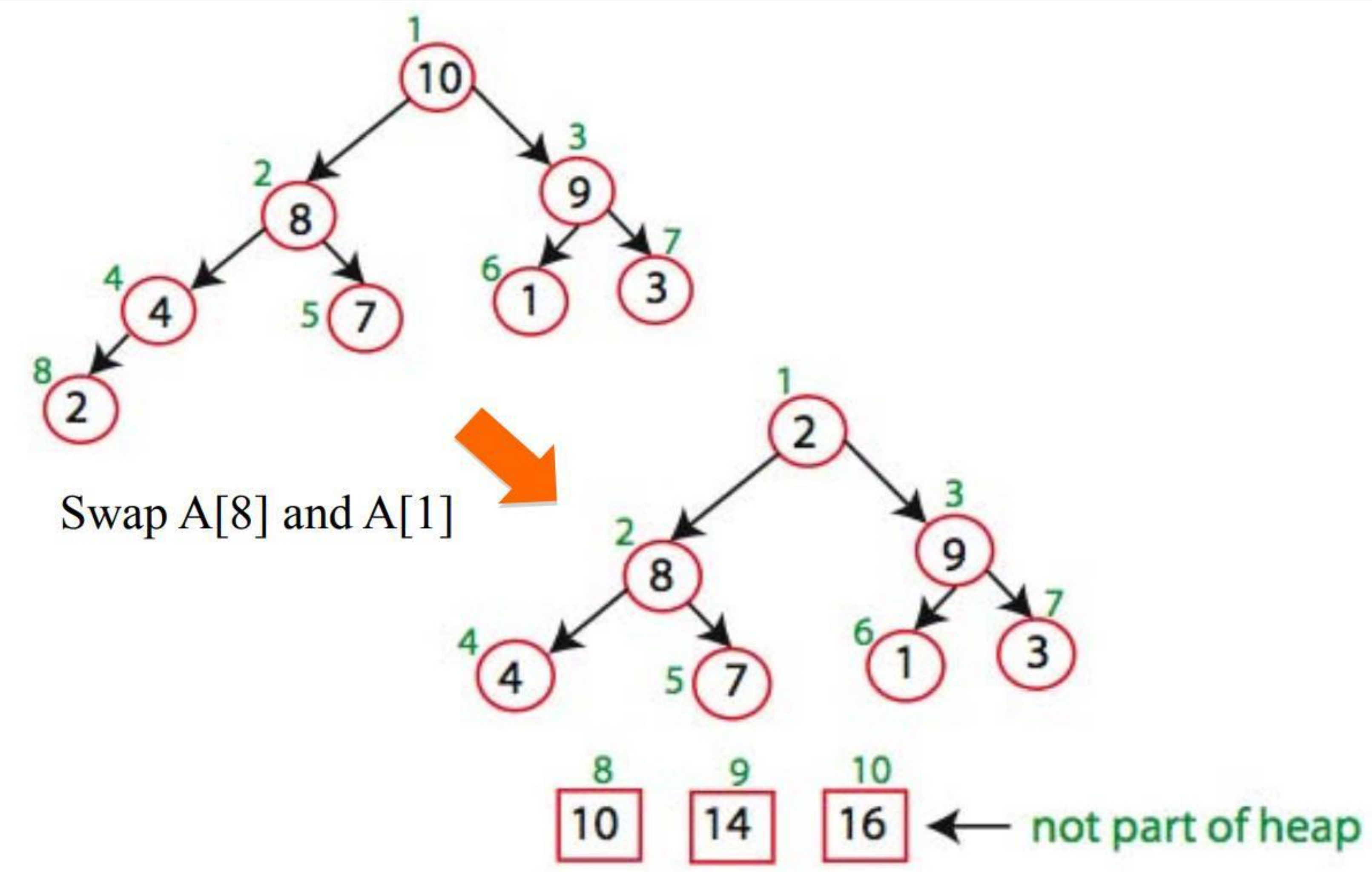

# Heapsort Time Analysis

- After $n$ iterations the Heap is empty
- Every iteration involves a swap and a max_heapify operation;
- Hence it takes $O(n \log n)$ time overall

# Divide and Conquer

- Very important technique in algorithm design
  - Divide problem into smaller parts
  - Independently solve the simpler parts
    - Think recursion
    - Or potential parallelism
  - Combine solution of parts to produce overall solution
- Two great sorting methods are fundamentally Divide-and-Conquer:
  - Merge Sort
  - Quick Sort

# Merge Sort

- So simple — really, soooooo simple

- Split the array into two halves
  - Sort (using the same merge sort) the first half
  - Then, sort the second half
  - Then, merge them (since they are ordered sequence, it should be easy to merge them in linear time into a single ordered sequence)

# Merge Sort

- Merging two sorted sequences into a single sorted sequence (in linear time):

  How to merge?

- Example:
  - Sequence A: 11, 23, 40, 57, 78, 93
  - Sequence B: 5, 9, 35, 36, 39, 63

# Merge Sort

- Sorting 38, 27, 43, 3, 9, 82, 10 using merging sort

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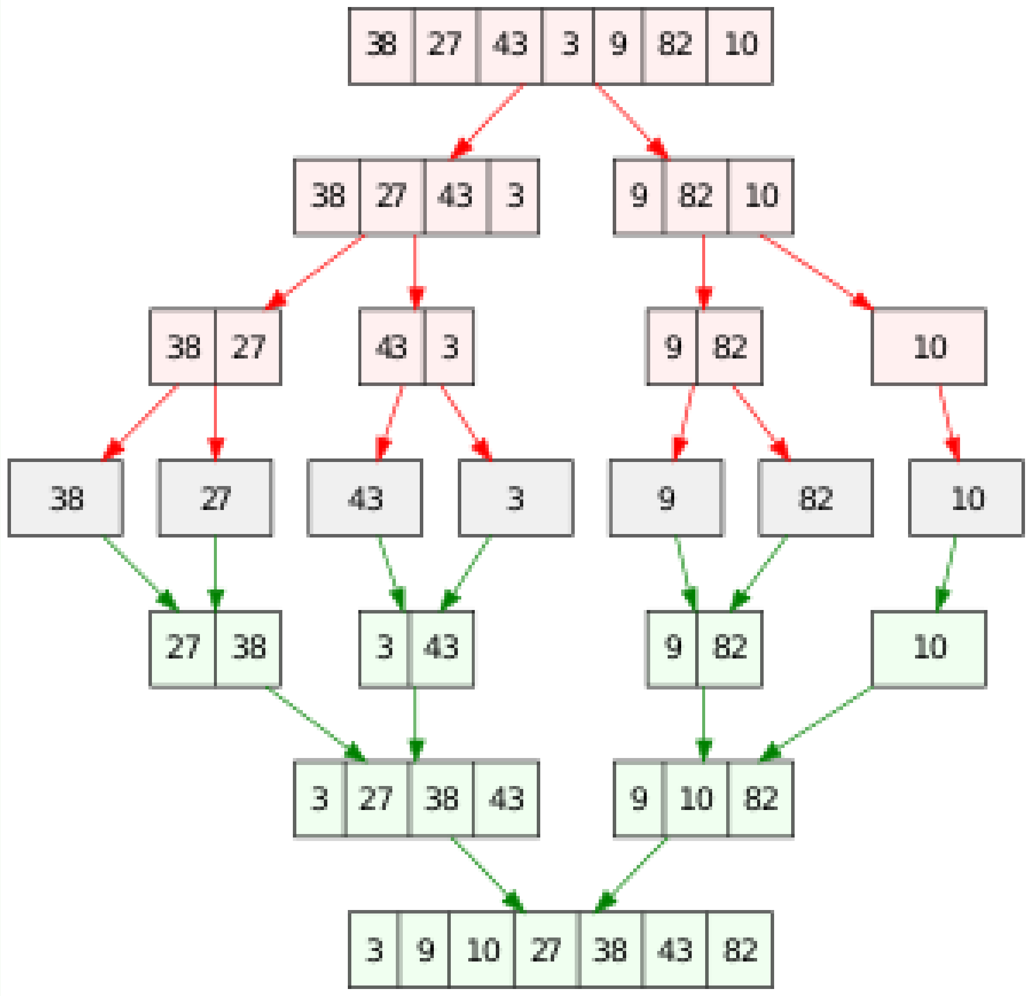


# Merge Sort

```
algorithm mergesort(A, left, right):
    if left < right then
        mid = ⌊(left + right) / 2⌋
        mergesort(A, left, mid)
        mergesort(A, mid + 1, right)
        merge(A, left, mid, right)
```

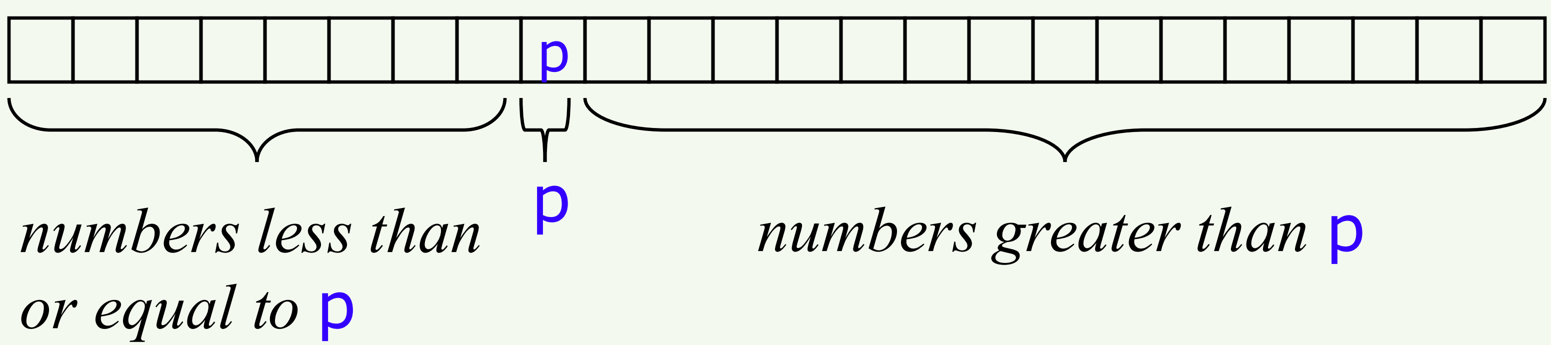
```
algorithm merge(A, left, mid, right):
    L ← A[left .. mid]
    R ← A[mid+1 .. right]
    i ← 1, j ← 1, k ← left

    while i ≤ length(L) and j ≤ length(R) do
        if L[i] ≤ R[j] then                  // Compare and copy
            A[k] ← L[i];  i ← i + 1
        else
            A[k] ← R[j];  j ← j + 1
        k ← k + 1

    while i ≤ length(L):                     // Append remaining L
        A[k] ← L[i];  i ← i + 1;  k ← k + 1
    while j ≤ length(R):                     // Append remaining R
        A[k] ← R[j];  j ← j + 1;  k ← k + 1
```

# Quick Sort

- Pick a "pivot" p (the pivot is a number in the list)
- Divide list into two sublists
  - One less-than-or-equal-to pivot value
  - One greater than pivot value
- Sort each sub-problem recursively
- Answer is the concatenation of the two solutions



# Quick Sort Pseudocode

```
algorithm quicksort(A, lo, hi):
    if lo < hi then
        p = partition(A, lo, hi)
        quicksort(A, lo, p - 1)
        quicksort(A, p + 1, hi)
```

First element is the pivot

```
algorithm partition(A, lo, hi):
    pivot = A[lo]
    i = lo
    j = hi + 1
    loop forever
        do
            i = i + 1
        while A[i] < pivot
        do
            j = j - 1
        while A[j] > pivot
        if i >= j then
            break
        else
            swap A[i] with A[j]
    swap A[j] with A[lo]
    return j
```

# Quick Sort: Example

- Sorting 7, 2, 8, 3, 5, 9, and 6 using quick sort

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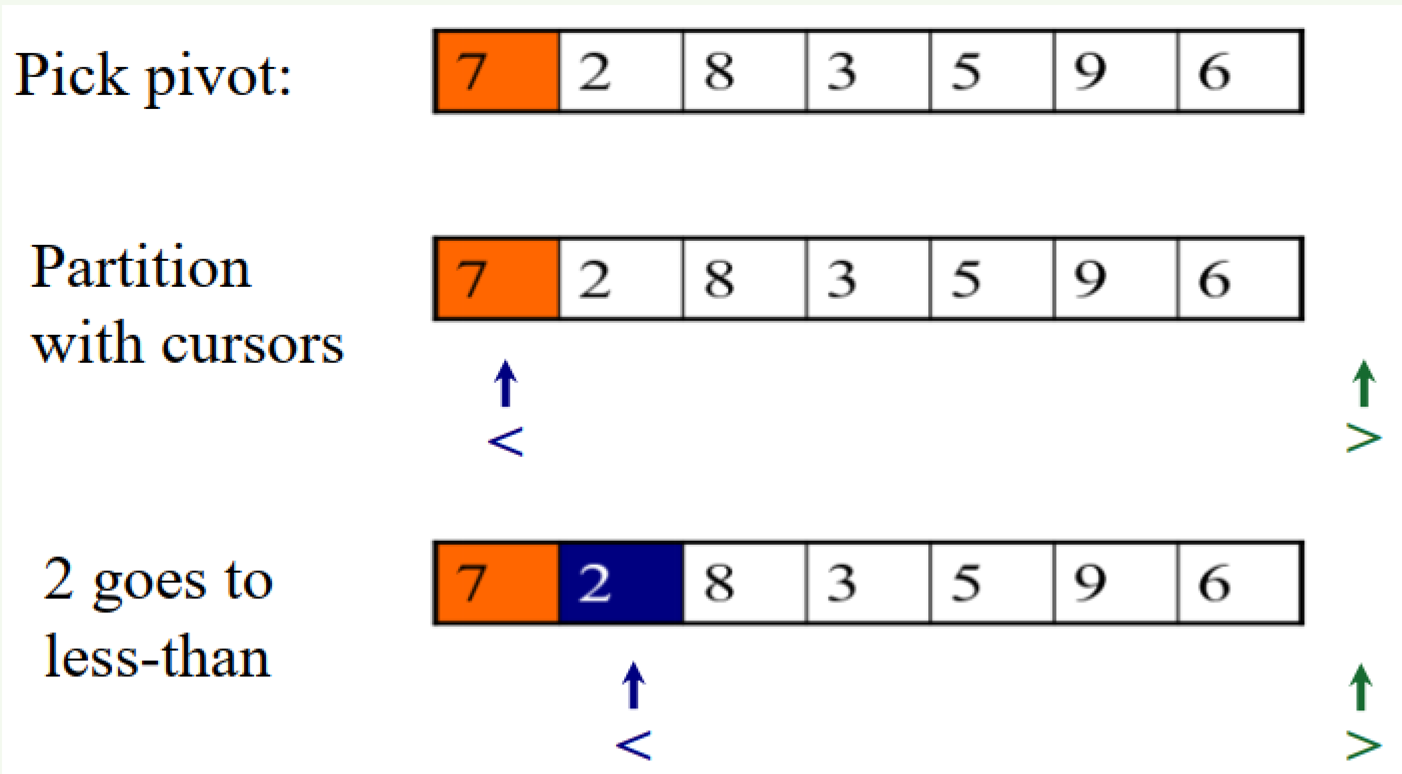

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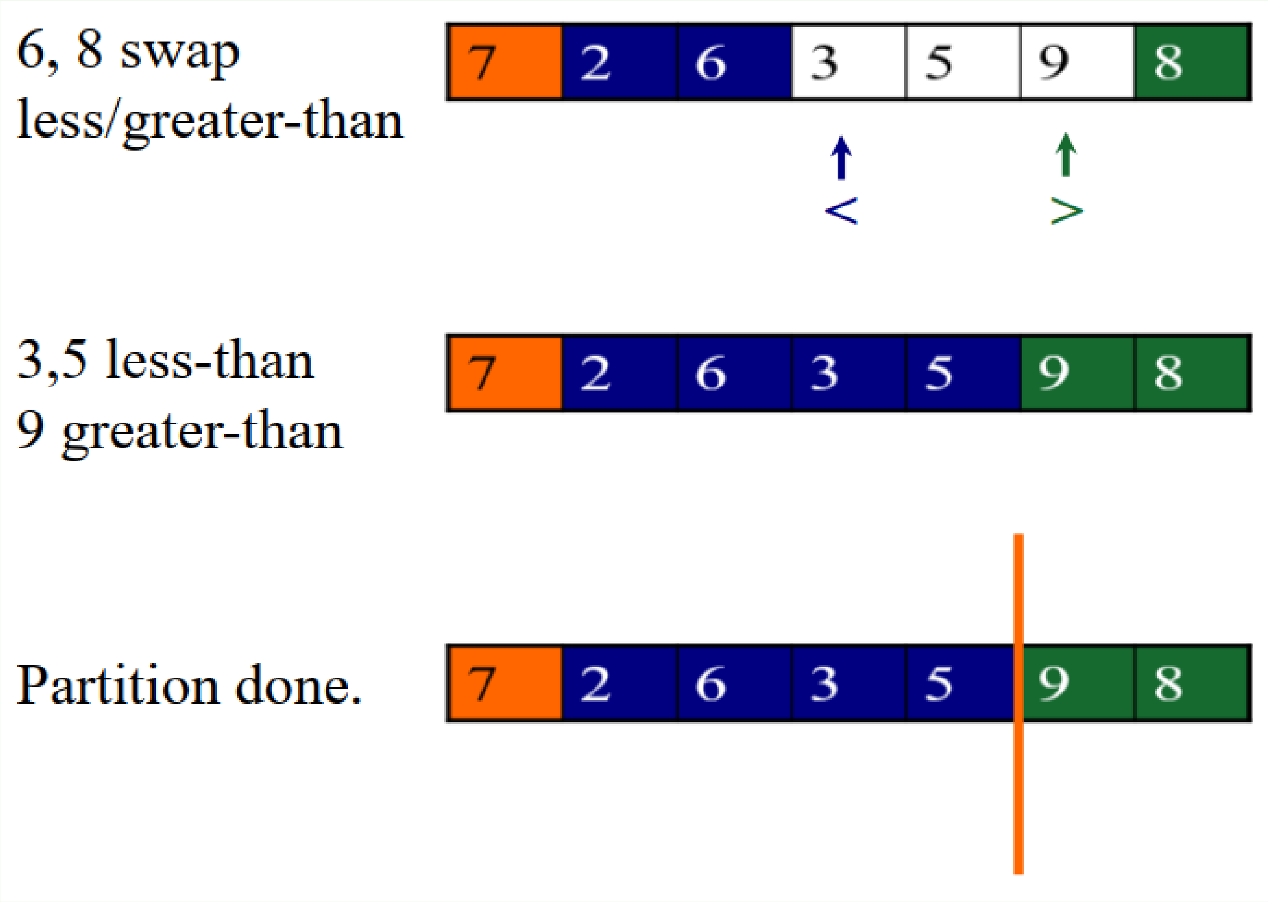
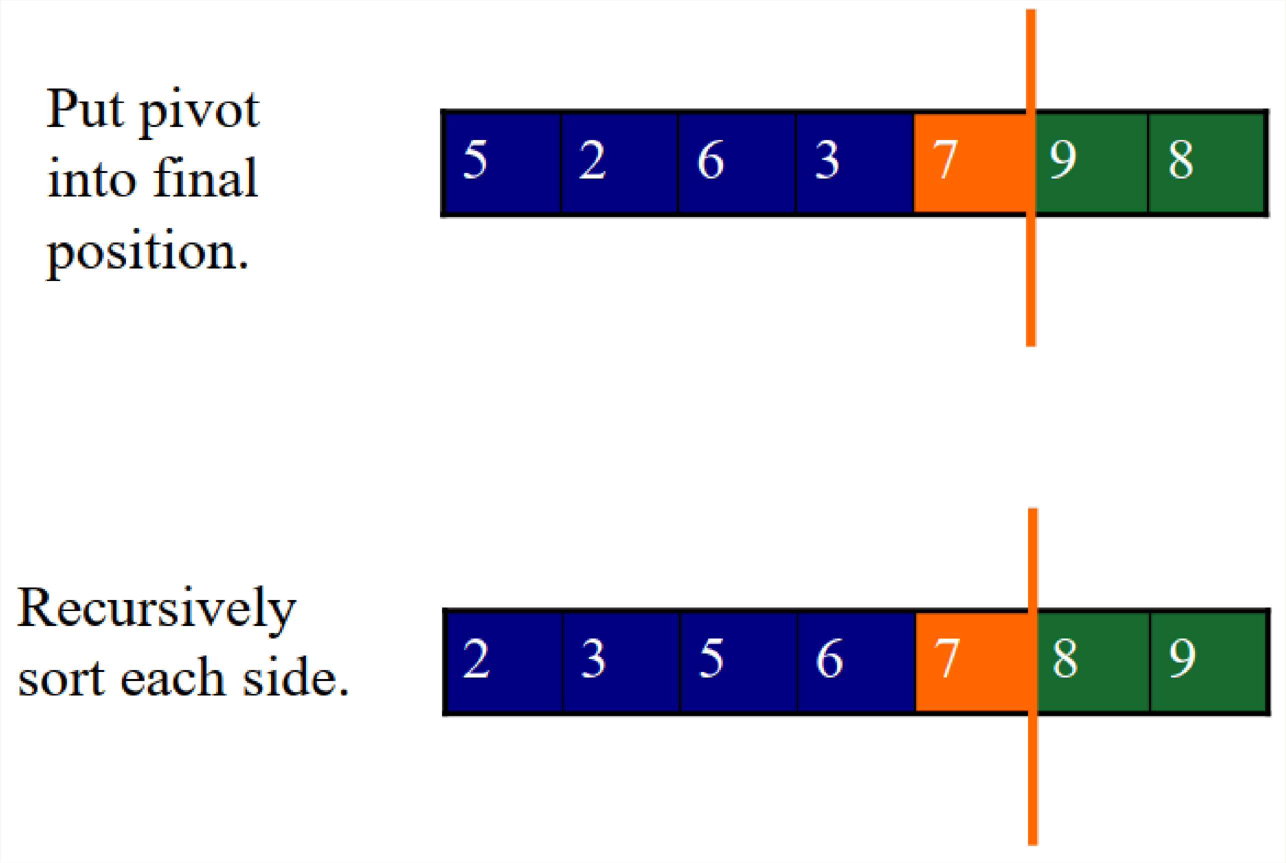


# Quick Sort: Example

- Sorting 7, 2, 8, 3, 5, 9, and 6 using merging sort



# Quick Sort: Example

- Partitioning on

  **4 3 6 9 2 4 3 1 2 1 8 9 3 5 6**

```
algorithm quicksort(A, lo, hi):
    if lo < hi then
        p = partition(A, lo, hi)
        quicksort(A, lo, p - 1)
        quicksort(A, p + 1, hi)
```

```
algorithm partition(A, lo, hi):
    pivot = A[lo]
    i = lo
    j = hi + 1
    loop forever
        do
            i = i + 1
        while A[i] < pivot
        do
            j = j - 1
        while A[j] > pivot
        if i >= j then
            break
        else
            swap A[i] with A[j]
    swap A[j] with A[lo]
    return j
```

# Quick Sort: Example

- Partitioning on

  **4 3 6 9 2 4 3 1 2 1 8 9 3 5 6**

- choose pivot: **4 3 6 9 2 4 3 1 2 1 8 9 3 5 6**
- search: **4 3 6 9 2 4 3 1 2 1 8 9 3 5 6**
- swap: **4 3 3 9 2 4 3 1 2 1 8 9 6 5 6**
- search: **4 3 3 9 2 4 3 1 2 1 8 9 6 5 6**
- swap: **4 3 3 1 2 4 3 1 2 9 8 9 6 5 6**
- search: **4 3 3 1 2 7 3 1 2 9 8 9 6 5 6**
- swap: **4 3 3 1 2 2 3 1 7 9 8 9 6 5 6**
- search: **4 3 3 1 2 2 3 1 7 9 8 9 6 5 6** (left > right)
- swap with pivot: **1 3 3 1 2 2 3 4 7 9 8 9 6 5 6**

# Quick Sort Time Analysis

- Picking pivot: constant time
- Partitioning: linear time
- Recursion: time for sorting left partition (say of size $i$) + time for right (size $N - i - 1$) + partition time

$$T(N)=T(i)+T(N - i - 1)+ cN$$

where $i$ is the number of elements smaller than the pivot

# Quick Sort Worst Case

- Quick Sort is fast in practice but has $\theta(N^2)$ worst-case complexity

- Pivot is always smallest element, so $i$ = 0:

$$T(N) = T(i) + T(N - i - 1) + cN$$
$$= T(N - 1) + cN$$
$$= T(N-2) + c(N-1) + cN$$
$$= T(N-k) + c\sum_{i=0}^{k-1}(N-i)$$
$$= O(N^2)$$

# Quick Sort Best Case

- Pivot is always middle element

$$T(N) = T(i) + T(N - i - 1) + cN$$

$$T(N) = 2T\left(\frac{N-1}{2}\right) + cN$$

$$< 2T\left(\frac{N}{2}\right) + cN$$

$$< 4T\left(\frac{N}{4}\right) + c\left(2\frac{N}{2} + N\right)$$

$$< 8T\left(\frac{N}{8}\right) + cN(1 + 1 + 1)$$

$$< kT\left(\frac{N}{k}\right) + cN\log k = O(N\log N)$$

# Dealing with Slow Quick Sort

- Randomly choose pivot
  - Good theoretically and practically, but call to random number generator can be expensive

- Pick pivot cleverly
  - "Median-of-3" rule takes Median(first, middle, last element elements) as pivot. Also works well
    - e.g., Swap Median with either first or last element, then partition as usual

# Integer sorting

- We've already discussed that (under some more or less standard assumptions), no sort algorithm can have a run time better than $n\log n$

- However, there are algorithms that run in linear time (huh???)

# Bucket Sort

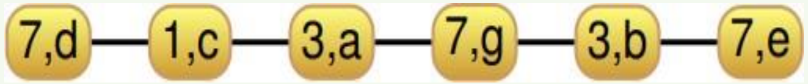
- If all keys are 0 … $K$ – 1
- Have an array of $K$ buckets (linked lists)
- Put keys into correct bucket of array
  - linear time!

- Bucket Sort is a stable sorting algorithm:
  - Items in input with the same key end up in the same order as when they began

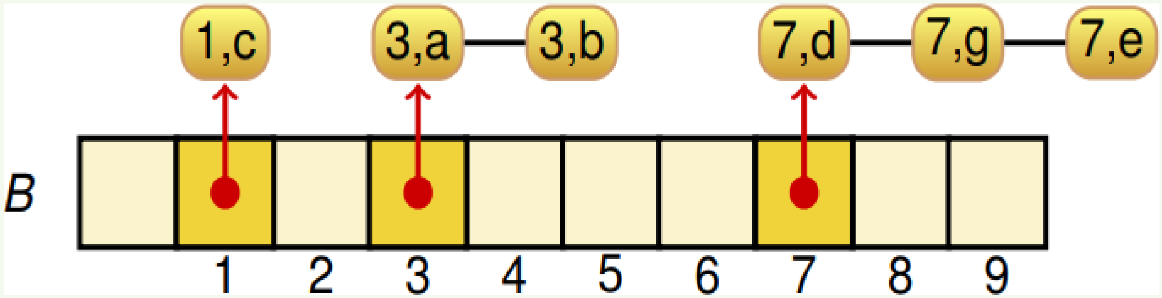
- Impractical for large $K$

# Bucket Sort: Example

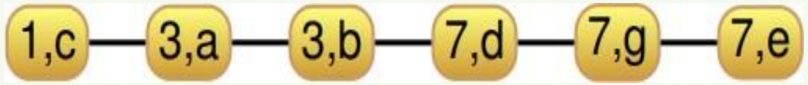
- Key range [0, 9]




- Phase 1: filling the buckets




- Phase 2: emptying the buckets into the list



# Bucket Sort Time Analysis

- Phase 1 takes $O(n)$ time
- Phase 2 takes $O(n + K)$ time
  - Thus bucket-sort is $O(n + K)$
- Very efficient if keys come from a small interval $[0, K - 1]$

# Radix Sort

- Radix = “The base of a number system” (Webster’s dictionary)
  - Alternate terminology: radix is number of bits needed to represent 0 to base 1; can say “base 8” or “radix 3”

- Idea: Bucket Sort on each digit, bottom up

# The Magic of Radix Sort

- Input list:

  126, 328, 636, 341, 416, 131, 328

- Bucket Sort on lower digit:

  341, 131, 126, 636, 416, 328, 328

- Bucket Sort result on next-higher digit:

  416, 126, 328, 328, 131, 636, 341

- Bucket Sort that result on highest digit:

  126, 131, 328, 328, 341, 416, 636

# Running Time of Radix sort

- $n$ items, $d$ digit keys
- How many passes?
- How much work per pass?
- Total time?

# Running Time of Radix sort

- $n$ items, $d$ digit keys
- How many passes? d
- How much work per pass? n

- Total time? O(dn)

# Summary

| | Best | Average | Worst |
|---|---|---|---|
| Insertion sort | $O(n)$ | $O(n^2)$ | $O(n^2)$ |
| Selection sort | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ |
| Heapsort | $O(n \log n)$ | $O(n \log n)$ | $O(n \log n)$ |
| Merge sort | $O(n \log n)$ | $O(n \log n)$ | $O(n \log n)$ |
| Quick sort | $O(n \log n)$ | $O(n \log n)$ | $O(n^2)$ |
| Bucket sort | $O(n + k)$ | $O(n + k)$ | $O(n + k)$ |
| Radix sort | $O(dn)$ | $O(dn)$ | $O(dn)$ |